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Effective Interactions in Dilute Mixtures of ^{3}He in ^{4}He

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Nonlocal pseudopotentials which describe the effective interaction between ^{3}He quasiparticles, and between these quasiparticles and the background ^{4}He liquid, are obtained as a function of concentration and pressure by generalizing the Aldrich–Pines pseudopotentials for pure ^{3}He and ^{4}He to dilute mixtures. The hierarchy of physical effects which determine these pseudopotentials is established. Interaction-induced short-range correlations are the dominant physical feature; next in order of importance is the greater zero point motion associated with the replacement of a ^{4}He atom by a ^{3}He atom, while spin-induced "Pauli principle" correlations play a significantly smaller, albeit still important role. We find a consistent trend in the change of the effective direct quasiparticle interactions with increasing concentration, and show how the Aldrich–Pines pseudopotentials for pure ^{3}He quasiparticles represent a natural extension of our results for dilute mixtures. Our calculated nonlocal pseudopotential for ^{3}He quasiparticles is qualitatively similar to that proposed by Bardeen, Baym, and Pines; it changes sign at somewhat lower momentum transfers than the BBP result, varies little with concentration, and provides a physical basis for understanding the BBP result. The effective interaction between quasiparticles of parallel spin, here determined for the first time, is essentially repulsive in the very dilute limit; as the concentration increases, it becomes increasingly attractive at low momentum transfers, and resembles closely that between antiparallel spin quasiparticles at 5% concentration. The concentration-dependent transport properties calculated from these pseudopotentials (which involve only one phenomenological parameter) are in good agreement with experiment at saturated vapor pressure (SVP), 10 atm, and 20 atm. Maxima in the thermal conductivity and spin diffusion are predicted to occur at concetrations somewhat less than 4%. Because the effective quasiparticle interactions are somewhat more repulsive than those previously proposed, we find the transition of the ^{3}He quasiparticles to the superfluid state takes

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place at significantly lower temperatures than many previous estimates; our predicted maximum superfluid transition temperature is 2×10^{-8} K (for a 0.6% mixture at 20 atm).

KEY WORDS: Dilute ^{3}He-^{4}He solutions; quantum liquids.

1. INTRODUCTION

The study of dilute solutions of ^{3}He in superfluid ^{4}He received its initial impetus from the experimental dicovery by Edwards *et al.* that at saturated vapor pressure (SVP) ^{3}He is miscible in ^{4}He, at concentrations less than about 6%, down to a temperature of absolute zero.(1) The thermal and transport properties of dilute solutions at low temperatures result from the interplay of the ^{3}He quasiparticles and the phonon–roton excitations of the superfluid. At very low temperatures the latter play a negligible role; the system then behaves as a Fermi liquid of ^{3}He quasiparticles moving through a background ^{4}He ether. By changing the ^{3}He concentration one can adjust the Fermi energy; hence mixtures provide an opportunity to study Fermi liquid effects over a wide range of densities.

Early experiments on the transport properties of ^{3}He–^{4}He mixtures were carried out by Anderson *et al.*,(2) while the pioneering theoretical calculations were carried out by Bardeen, Baym, and Pines (BBP),(3) with results which agreed well with experiment. BBP showed that the effective interaction between two ^{3}He quasiparticles is made up of two distinct contributions: a direct interaction and a ^{4}He-induced interaction. Both interactions are of the order $\sim m_4 s^2$ in the long-wavelength limit, while the combination of the yields a rather weak interaction $\sim(\alpha^2 m_4 s^2)$, where s is the ^{4}He sound velocity and $\alpha \simeq 0.28$ is the excess volume occupied by the ^{3}He impurity. The physical origin of the weakness of this interaction comes from the fact that ^{3}He is an isotopic impurity of ^{4}He. Since the force fields of the ^{3}He and ^{4}He atoms are identical, two ^{3}He atoms in a ^{4}He background "feel" each other only because they occupy a larger volume. The long-wavelength part of the effective interaction between two ^{3}He quasiparticles is thus equivalent to that between two "holes" of relative volume α in the liquid.

BBP further assumed that the effective interaction depends only on the momentum difference between the initial and final states of two ^{3}He quasiparticles, and is independent of the ^{3}He concentration. In order to explain the results of spin diffusion experiments, they proposed a simple phenomenological q-dependent effective interaction,

$$V^{\mathrm{BBP}}(q) = V_0^{\mathrm{BBP}} \cos(\beta q) \tag{1.1}$$

where $V_0^{\mathrm{BBP}} \approx -\alpha^2 m_4 s^2/n_4$ and their best fit value of β was found to be $\cong 3.16$ Å. BBP used this model potential to calculate the Fermi liquid parameters and the ^{3}He superfluid transition temperature T_c. They found that $T_c \simeq 10^{-6}$ °K for a 1.3% solution, and $T_c \simeq 10^{-9}$ °K for a 5% solution. Their calculated result for the thermal conductivity, κ, however, did not yield a consistent fit to the subsequent measurement$^{(4)}$ at either $x = 1.3\%$ or 5% concentrations.

Following the pioneering work of BBP, several theories have been proposed to give a better account of transport measurements:

(i) Ebner$^{(5)}$ constructed an effective interaction $V(q)$ in the form of a power series in q^2. By assuming $V(q)$ is independent of concentration and fitting his results for κ and D to experiment, Ebner determined the coefficients in the power series. Ebner's approach can thus be regarded as an improved treatment of the momentum dependence of the effective interaction; his resulting potential is not far from that proposed by BBP.

(ii) Fu and Pethick$^{(6)}$ studied the concentration dependence of the transport properties by adopting the BBP potential, Eq. (1.1), as a bare interaction between two ^{3}He particles, and then using perturbation theory to calculate the scattering amplitudes as functions of ^{3}He concentration, x. They showed that although the strength of the two-particle interaction, (1.1), is weak, the expansion parameter in perturbation theory, $N(0)V_0^{\mathrm{BBP}}$ [$N(0)$ is the density of states], is large ($\simeq 0.3$ for $x = 1.3\%$). This approach leads to quite good agreement between theory and experiment for κ and the viscosity η, but is less successful in yielding agreement with the experiments of Murdock *et al.*$^{(7)}$ for the spin diffusion in 5% mixtures.

(iii) Bashkin$^{(8)}$ calculated the thermodynamic and transport quantities of the ^{3}He quasiparticles by treating these as a dilute Fermi gas with a short-range interaction. In this theory, all quantities of interest are expressed in terms of a single parameter—the s-wave scattering length. Bashkin's results provide an acceptable fit to experiment ar $x = 1.3\%$. However, Bashkin was led to predict a superfluid transition temperature, T_c, in the millidegree range, a prediction which subsequent experiments have shown is invalid.$^{(9)}$

From these results it is evident that both the momentum and concentration dependence of the effective interaction between ^{3}He quasiparticles play an important role in determining the transport properties and the superfluid transition temperature of dilute mixtures. Ideally one would like to know the momentum dependence of these effective interactions in the very dilute limit, and then examine the way in which changes in this interaction with increasing concentration influence both transport and possible superfluidity. However, the determination of this effective interaction is not an easy task, because the very considerable cancellation of the direct interaction against the induced interaction means that each quantity must be known to a very high degree of accuracy in order to arrive at a reasonable accurate result for their sum.

More specifically, as we shall see in the following sections, one needs to know four quantities in order to determine the effective interactions: the effective direct interaction between ^{3}He quasiparticles of parallel and antiparallel spin; the effective interaction between a ^{3}He quasiparticle and the background ^{4}He liquid, and the nonlocal (i.e., wave-vector-dependent) static density susceptibility of the ^{4}He background. In this paper we determine these quantities by taking as our starting point the theory developed by Aldrich and Pines (hereafter AP) for the calculation of elementary excitations in liquid ^{4}He(10) and ^{3}He.(11,12),[2] Their theory yields directly the nonlocal ^{4}He density susceptibility in the very low concentration limit, and can easily be extended to obtain this quantity as a function of ^{3}He concentration. AP obtain the restoring force for zero sound in both ^{4}He and ^{3}He from a configuration space pseudopotential which describes in simple physical fashion the way in which the effective interaction between quasiparticles in the liquid differs from that of bare ^{4}He or ^{3}He atoms. We adopt a similar approach to the calculation of the effective direct ^{3}He–^{3}He quasiparticle interactions and the ^{3}He–^{4}He interaction in dilute mixtures. At any given system density we take the AP configuration space interaction, $f^s(r)$, between ^{4}He quasiparticles as our benchmark, and develop a simple physical model for the way in which this interaction changes when one replaces one or both of the ^{4}He particles by a ^{3}He quasiparticle.

Dilute mixtures may be regarded as providing a bridge between pure ^{4}He and ^{3}He liquids; as one replaces ^{4}He atoms with ^{3}He atoms, one can study the way in which the isotopic mass change (which gives rise to an enhancement in the liquid of the zero point motion of the ^{3}He atoms compared to ^{4}He) and the Pauli principle (which influences the short-range correlations between ^{3}He quasiparticles of parallel spin) act to alter the very strong short-range correlations between ^{4}He atoms in the liquid which are responsible for the difference between $f^s(r)$ and the bare two-body interaction. We show that there is a well-defined hierarchy of physical effects: the strong short-range particle interactions play the dominant physical role; next comes the influence of zero point motion, while least important, but still of substantial physical importance for the properties of either dilute mixtures or pure ^{3}He, are the effects of the Pauli principle. We find moreover a consistent trend in the change of the effective direct interactions with increasing concentration, and show how the AP form for

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[2] For a recent review of the polarization potential approach to quantum liquids, see Ref. 12.

the effective quasiparticle interactions in pure ^{3}He represents a natural extension of our results for dilute mixtures.

We use our calculated model pseudopotentials to construct net effective interactions between ^{3}He quasiparticles of parallel and antiparallel spin. In the very dilute limit at SVP the latter effective interaction resembles that proposed by BBP and Ebner,$^{(5)}$ in that it changes sign from attraction to repulsion at momentum transfers $\sim 0.4\,\text{Å}^{-1}$; the precise form of our interaction is, however, somewhat different. From the concentration and pressure dependence of our pseudopotentials we are able to calculate explicitly the changes with concentration and pressure of the net ^{3}He quasiparticle interactions. The latter may then be used to calculate the corresponding scattering amplitudes, and through these, transport and superfluid properties.

Because detailed experimental information on the Landau parameters as a function of concentration is not available for dilute mixtures, our model pseudopotentials are constructed with the aid of simple scaling arguments for the concentration dependence of the spatial averages of the configuration space interactions between ^{3}He quasiparticles and the background ^{4}He liquid, and for the direct effective interactions between ^{3}He quasiparticles. Our theory contains only one free parameter (which determines the configuration space interaction between parallel spin ^{3}He quasiparticles); this parameter is chosen to provide the best fit to spin diffusion experiments.

There are three kinds of experimental tests of the effective interactions we derive in this paper: transport properties (spin diffusion, thermal conductivity, viscosity); the current upper limit on the superfluid transition temperature; and neutron scattering experiments on elementary excitations in dilute mixtures,$^{(13,14)}$ which provide information both on the way in which the phonon–maxon–roton spectrum is altered, and on the density and spin-density fluctuation excitation spectrum of the ^{3}He quasiparticles. In the present paper we confine our attention to the first two experimental tests; we report on the results of calculations which show good agreement between theory and experiment for mixtures at SVP, 10 atm, and 20 atm. We predict a maximum superfluid transition temperature of some 2×10^{-8} K (for a 0.6% mixture at 20 atm), a prediction which may prove easier to disprove than verify. We show elsewhere$^{(15)}$ that agreement between theory and experiment is equally satisfactory for the results of the neutron scattering experiments.

The plan of this paper is the following. In Section 2 we consider the extension to finite momentum transfers of the BBP approach to the effective quasiparticle interaction and review briefly polarization potential theory. In Section 3, following a review of the AP configuration space pseudopotentials for pure ^{4}He and ^{3}He, we construct pseudopotentials to describe the

^{3}He–^{4}He and ^{3}He–^{3}He direct effective interactions, and combine these with our results for the nonlocal density susceptibility to obtain the net concentration and momentum-dependent effective interactions between ^{3}He quasiparticles. In Section 4 we report on the results of our calculations of the transport properties and maximum superfluid transition temperature for mixtures of arbitrary concentration at SVP, 10 atm, and 20 atm, while Section 5 contains our summary and conclusions.

2. DIRECT AND INDUCED QUASIPARTICLE INTERACTION

The long-wavelength macroscopic approach of BBP to the calculation of the ^{3}He quasiparticle interaction may be extended to finite momentum transfers in the following way. Let the *direct* effective interaction between ^{3}He quasiparticles of parallel and antiparallel spin be described by momentum-dependent interactions, $V_q^{\uparrow\uparrow}$ and $V_q^{\uparrow\downarrow}$, while the effective interaction between these quasiparticles at positions $\mathbf{r}_i$ and the background ^{4}He liquid takes the form

$$H_{3\text{-}4}=\sum_{qi} u_q \rho_q^+ e^{i\mathbf{q}\cdot\mathbf{r}_i} \tag{2.1}$$

where ρ_q is the ^{4}He density fluctuation. In this paper we shall confine our attention to low-frequency properties of the ^{3}He quasiparticles, so that we need not take into account their coupling to current fluctuations in the background ^{4}He liquid; it is then the interaction, Eq. (2.1), which is responsible for the induced interaction between the ^{3}He quasiparticles. The latter may be simple derived without restoring to field theoretic methods by using linear response theory.(16) In this approximation (which is applicable to dilute mixtures), a given ^{3}He quasiparticle, at $\mathbf{R}_i$, say, induces a static ^{4}He density fluctuation,

$$\langle \rho_q \rangle = \chi(q,0) e^{i\mathbf{q}\cdot\mathbf{R}_i} \tag{2.2}$$

where $\chi(q,0)$ is the nonlocal static density susceptibility (or static density–density response function) of the background ^{4}He liquid. The action of this density fluctuation on a quasiparticle at $\mathbf{R}_j$ is, according to Eq. (2.1),

$$u_q^2 \chi(q,0) e^{iq\cdot\mathbf{R}_i-\mathbf{R}_j} \tag{2.3}$$

It follows that the net interaction between ^{3}He quasiparticles, $f_q^{\uparrow\uparrow}$ or $f_q^{\uparrow\downarrow}$, takes the form

$$f_q^{\uparrow\uparrow}=V_q^{\uparrow\uparrow}+u_q^2\chi(q,0) \tag{2.4a}$$

$$f_q^{\uparrow\downarrow}=V_q^{\uparrow\downarrow}+u_q^2\chi(q,0) \tag{2.4b}$$

In practice one is often interested in the spin-symmetric and spin-antisymmetric quasiparticle interestions, which are defined as

$$f_q^s = (f_q^{\uparrow\uparrow} + f_q^{\uparrow\downarrow})/2 \tag{2.5a}$$

$$f_q^a = (f_q^{\uparrow\uparrow} - f_q^{\uparrow\downarrow})/2 \tag{2.5b}$$

Since the induced interaction is spin independent it follows that f_q^a is determined entirely by the spin dependence of the direct effective interactions between the ^{3}He quasiparticles.

All four quantities which enter into the determination of $f_q^{\uparrow\downarrow}$ and $f_q^{\uparrow\downarrow}$ may be expected to vary with concentration. One does not however have complete freedom in choosing these quantities, since in the long-wavelength limit $\chi(q, 0)$ must reduce to the compressibility; thus

$$\lim_{q\to 0} \chi(q, 0) = -\frac{n(x)}{m_4 s^2(x)} \tag{2.6}$$

where s and n are the first sound velocity and density in the mixture. Moreover, the quantity

$$f_0^a \equiv V_0^{\uparrow\uparrow} - V_0^{\uparrow\downarrow} \tag{2.7}$$

is related to the Landau theory parameter F_0^a, and the experimentally measured spin susceptibility, χ_P of the mixture, according to

$$F_0^a = N(0) f_0^a \tag{2.8}$$

and

$$\chi_P = \frac{m^* p_F}{\pi^2 \hbar^3} \frac{\beta^2}{1 + F_0^a} \tag{2.9}$$

where $N(0)$ is the density of states per unit energy, m^* the quasiparticle effective mass, and β the Bohr magneton. Similarly, f_0^s is related to the spin-symmetric dimensionless Landau parameter,

$$F_0^s = N(0) f_0^s \tag{2.10}$$

However, as discussed in the following section it is a present difficult to extract the latter quantity from experiments. Finally, in the low concentration limit one knows from the work of BBP the quantities u_0 and $V_0^{\uparrow\downarrow}$, while we show in the next section that in this limit one likewise can express $V_0^{\uparrow\uparrow}$ in terms of α, the excess molar volume.

We determine the quantities $V_q^{\uparrow\uparrow}$, $V_q^{\uparrow\downarrow}$, and u_q by constructing pseudopotentials which are closely related to the pseudopotential, f_q^s, which

Aldrich and Pines (AP) used to describe the effective interaction between ^{4}He quasiparticles in liquid ^{4}He.$^{(10)}$ In their polarization potential theory of the helium liquids, AP describe the consequences of the strong interactions in ^{3}He and ^{4}He in terms of self-consistent fields. The strengths of these fields are determined by physical arguments, static measurements, and sum rule considerations. The theory makes possible a unified treatment of the elementary excitation spectra of both ^{3}He and ^{4}He, and yields results in excellent agreement with experiment.

For ^{4}He, the AP self-consistent fields are a scalar polarization potential

$$f_q^s\langle\rho(q,\omega)\rangle \tag{2.11}$$

which couples to the density fluctuations, $\langle\rho(q,\omega)\rangle$, and a vector potential

$$f_q^v\langle\mathbf{J}(q,\omega)\rangle \tag{2.12}$$

which couples to the current fluctuations, $(\mathbf{J}(q,\omega)\rangle$, and takes backflow into account. The ^{4}He density–density direct response function is then calculated by considering the response of the system to an external field plus these self-consistent fields; in the static limit of interest to us here, one can neglect the backflow term, Eq. (2.12), and write

$$\chi(q,0)=\frac{\chi_{44,\mathrm{sc}}(q,0)}{1-f_q^s\chi_{44,\mathrm{sc}}(q,0)} \tag{2.13}$$

where $\chi_{44,\mathrm{sc}}(q,0)$ is the linear response of the system density to the sum of an external field and the internal polarization field, Eq. (2.11) coupled to the ^{4}He density fluctuations. It is the strength of the internal polarization field, f_q^s, which is of central interest to us in this paper. [We refer the interested reader to AP$^{(10)}$ and to Ref. 17 for a discussion of the calculation of $\chi_{\mathrm{sc}}(q,0)$.]

3. ^{3}He–^{4}He AND ^{3}He–^{3}He EFFECTIVE INTERACTIONS

Before proceeding to calculate the pseudopotentials which describe ^{3}He-^{4}He and ^{3}He-^{3}He interactions in dilute mixtures it is instructive to review briefly the results which have been obtained for pure ^{4}He and pure ^{3}He.

3.1. Pseudopotentials for ^{4}He and ^{3}He

Aldrich and Pines$^{(10,11)}$ developed a physical model for the nonlocal restoring forces responsible for the collective modes in ^{4}He and ^{3}He by relating their strengths to configuration space pseudopotentials which describe the effective interaction between quasiparticles. In the case of ^{4}He, they argued that on going from a gas to a liquid, the major change in the effective interaction between ^{4}He atoms will come at short distances where the strong (nearly hard core) bare atom repulsion makes it unlikely that the particles will approach sufficiently close to feel the full consequences of that repulsion. Thus they argued that the long-range part of the configuration space effective interaction, $f^s(r)$, will be nearly the same as that between bare ^{4}He atoms, while the short-range part (i.e., at $r \lesssim 2.68$ Å, the distance at which the bare interaction changes from attraction to repulsion) will be altered to a soft core repulsion. They took $f^s(r)$ to have the functional form

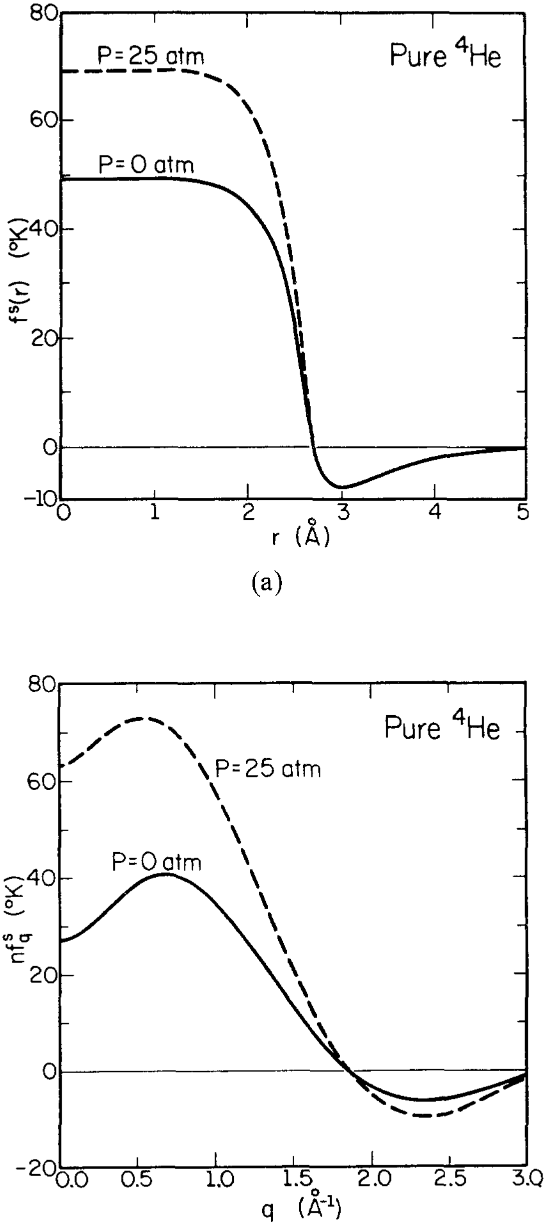
$$f^s(r) = \begin{cases} a\left[1 - \left(\frac{r}{r_c}\right)^8\right], & r < r_c \\ b\left[\left(\frac{r_c}{r}\right)^{12} - \left(\frac{r_c}{r}\right)^6\right], & r_c < r < r_t \\ -\left(\frac{a_8}{r^8} + \frac{a_6}{r^6}\right), & r_t < r \end{cases} \tag{3.1}$$

where $a_8 = 28400.5$ Å^8, $a_6 = 10163.3$ Å^6, $r_t \cong 5$ Å, and the parameter b is chosen to yield a continuous curve at r_t. The spatial average of $f^s(r)$, f_0^s, is known from the compressibility sum rule, $f_0^s = m_4 s^2/n$. Hence once one has chosen the shape function for $f^s(r)$, as in Eq. (3.1), there is only one free parameter; this may be taken to be either the distance, r_c, at which $f^s(r)$ changes from being repulsive to attractive, or the core height at the origin, a.

AP chose r_c at all pressures to be identical to the range of the repulsive interaction between two bare helium atoms, $r_c = 2.68$ Å, and further assumed that the shape of the repulsive interaction did not vary with pressure. Hence at any density $f^s(r)$ is uniquely determined by the compressibility. Their results for $f^s(r)$ and its Fourier transform, f_q^s, at $P = 0$ and 25 atm are shown in Figs. 1a and 1b. It is this interaction between ^{4}He atoms which acts as a benchmark for the effective ^{3}He–^{4}He and ^{3}He–^{3}He interactions in the mixtures. Note that since the strength of the interaction, $f_0^s = m_4 s^2/n$, increases with pressure the repulsive part of the pseudopotential likewise must increase, since the attractive part is unchanged.

For ^{3}He, two pseudopotentials need to be determined: $f^{\uparrow\uparrow}(r)$, which describes the interaction between two spin-parallel particles; and $f^{\uparrow\downarrow}(r)$, which describes the interaction between two spin-antiparallel particles. AP$^{(11)}$ took the r dependence of these interactions to be of the same form as Eq. (3.1); the spin-symmetric and spin-antisymmetric potentials $f^s(r)$ and $f^a(r)$ are then the averages of the sum and difference of $f^{\uparrow\uparrow}(r)$ and $f^{\uparrow\downarrow}(r)$,

$$f^{s,a}(r) = \tfrac{1}{2}[f^{\uparrow\uparrow}(r) +, - f^{\uparrow\downarrow}(r)] \tag{3.2}$$



(a)

(b)

Fig. 1. (a) The Aldrich–Pines pseudopotential $f^s(r)$ for ^{4}He at SVP (solid curve) and at $P = 25$ atm (dashed curve). (b) The scalar polarization potential nf^s_q, for 0 and 25 atm.

while the long-wavelength limits f_0^s and f_0^a, of the Fourier transforms, f_q^s and f_q^a, are related to the measured Landau parameters, F_0^s and F_0^a, according to

$$f_0^s = F_0^s/N(0) \tag{3.3a}$$

$$f_0^a = F_0^a/N(0) \tag{3.3b}$$

where $N(0)$ is the density of states per unit energy.

It is instructive to compare the spatial averages, $f_0^{\uparrow\uparrow}(^3\mathrm{He})$ and $f_0^{\uparrow\downarrow}(^3\mathrm{He})$, of these pseudopotentials with the corresponding quantity for ^{4}He, f_0^s. ^{3}He at 21 atm has the same density, n, as ^{4}He at SVP, yet one finds that $nf_0^{\uparrow\downarrow}(^3\mathrm{He}) = 46.5$ K, $nf_0^{\uparrow\uparrow}(^3\mathrm{He}) = 45.2$ K, while $nf_0^s = 27.3$ K (see Table II). For particles of antiparallel spin, the Pauli principle plays no role in determining the effective interaction; the substantial difference between $f_0^{\uparrow\downarrow}(^3\mathrm{He})$ and f_0^s must therefore be attributed to the larger zero point motion of ^{3}He atoms. AP argued that this increase comes about because the zero point motion in the liquid increases the distance, $r^{\uparrow\downarrow}$, at which the effective interaction between ^{3}He atoms becomes attractive. They found excellent agreement with the neutron scattering measurements of the zero sound spectrum with $r^{\uparrow\downarrow} = 3$ Å for ^{3}He at SVP. From this point of view it is to be expected that at higher pressures the effect will be less pronounced, so that $r^{\uparrow\downarrow}$ should decrease with pressure. In recent work with Bedell, we find$^{(18)}$ that at 21 atm $r^{\uparrow\downarrow} \simeq 2.8$ Å, in which case, as may be seen on comparing Fig. 2

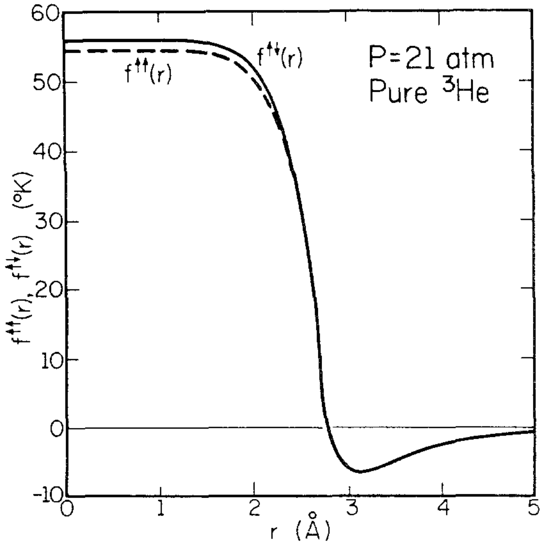


Fig. 2. The pseudopotentials $f^{\uparrow\uparrow}(r)$ (dashed curve) and $f^{\uparrow\downarrow}(r)$ (solid curve), for ^{3}He at $P = 21$ atm. The difference between $r^{\uparrow\uparrow}$ *and* $r^{\uparrow\downarrow}$ is too small to be seen on this scale.

with Fig. 1a, for ^{3}He and ^{4}He at the same density the enhanced zero point motion of the ^{3}He atoms not only acts to increase the effective range of the repulsive part of the interaction, but also reduces the effectiveness of the screening of this repulsive interaction (the core height $a^{\uparrow\downarrow} = 55.7$ K, while the core height for pure ^{4}He is 49.3 K).

What now is the role of the Pauli principle? Since $f_0^{\uparrow\uparrow}(^3\mathrm{He})$ is less than $f_0^{\uparrow\downarrow}(^3\mathrm{He})$, it is evident that while the Pauli principle acts to reduce slightly the overall strength of the repulsive interaction between quasiparticles, its effect is, in this case, much smaller than that of the enhanced zero point motion. One might expect on general grounds that the range, $r^{\uparrow\uparrow}$, of the repulsive part of the interaction between parallel spin quasiparticles will differ somewhat from $r^{\uparrow\downarrow}$. Aldrich and Pines noted that this difference,

$$\delta = r^{\uparrow\uparrow} - r^{\uparrow\downarrow} \tag{3.4}$$

was too small to be determined accurately from their fit to the neutron scattering experiments. Subsequently Bedell and Pines$^{(19)}$ found that the quasiparticle scattering amplitudes constructed from the AP pseudopotentials are very sensitive to the parameter δ; by comparing the calculated transport coefficients with experiments, they found that $\delta \cong 0.03$ Å at SVP and that it decreased with increasing pressure until it reached a constant value, $\delta \cong 0.007$ Å at $P \gtrsim 21$ atm. This "best fit" decrease in δ as the pressure increases confirms the argument of Aldrich and Pines that the role played by statistics becomes less inportant as the strength of interaction between particles becomes stronger. The results of BHP$^{(18)}$ for the peudopotentials $f^{\uparrow\uparrow}(r)$ and $f^{\uparrow\downarrow}(r)$ at $P = 21$ atm are shown in Fig. 2.

3.2. Effective Quasiparticle Interactions in ^{3}He–^{4}He Mixtures

To calculate f_q^s and f_q^a in ^{3}He–^{4}He mixtures we need to know four quantities: the effective ^{3}He–^{4}He quasiparticle interaction, u_q; the static density–density response function, $\chi(q, 0)$; and the effective direct interactions between ^{3}He quasiparticles of parallel and antiparallel spin, $V_q^{\uparrow\uparrow}$ and $V_q^{\uparrow\downarrow}$. We determine the three potentials as a function of concentration by using the same approach as that adopted by Aldrich and Pines for ^{3}He and ^{4}He: we assume that the long-range part of the interaction in configuration space is identical to that between bare atoms, that it goes over smoothly to a repulsive interaction at some distance r_c ($\gtrsim 2.68$ Å), and that it takes the form of a simple soft-core repulsion $a[1 - (r/r_c)^8]$ for $r \lesssim r_c$. With this assumption, for a given choice of r_c, the potential is uniquely determined by its spatial average. We therefore need to know

$$u_0 = \lim_{q\to 0} u_q \equiv \frac{4\pi}{V} \lim_{q\to 0} \int_0^\infty dr\, r^2 \frac{\sin qr}{qr} u(r) \tag{3.5}$$

$$V_0^{\uparrow\uparrow(\uparrow\downarrow)} = \lim_{q\to 0} \int_0^\infty dr\, r^2 \frac{\sin qr}{qr} V^{\uparrow\uparrow}(r)^{(\uparrow\downarrow)} \tag{3.6}$$

as a function of concentration, and then determine the corresponding quantities, r_c^{3-4}, $r_c^{\uparrow\downarrow}$, and $r_c^{\uparrow\uparrow}$.

3.3. $V_0^{\uparrow\uparrow}$, $V_0^{\uparrow\downarrow}$, u_0

The long-wavelength limit of the effective interaction between ^{4}He quasiparticles as a function of density (or ^{3}He concentration, x), is readily obtained from the measured value of the first sound velocity or compressibility:

$$f_{0,4}^s = \frac{ms^2(x)}{n(x)} = \frac{1}{n^2(x)\kappa} \tag{3.7}$$

For pure ^{3}He, the long-wavelength interactions between quasiparticles of parallel (antiparallel) spin are obtained in similar fashion from the experimentally measured specific heat, compressibility and spin susceptibility, χ_P, according to

$$N(0) f_0^s = F_0^s = \frac{N(0)}{n^2\kappa} - 1 \tag{3.8}$$

$$N(0) f_0^a = F_0^a = \frac{\hbar^2}{4} \frac{\gamma^2 N(0)}{\chi_P} - 1 \tag{3.9}$$

where γ is the gyromagnetic ratio, and $N(0)$, the density of states, is given by

$$N(0) = \frac{m^* q_F}{\pi^2 \hbar^3} \tag{3.10}$$

m^* is determined from the specific heat, and

$$f_0^{\uparrow\uparrow} = f_0^s + f_0^a \tag{3.11}$$

$$f_0^{\uparrow\downarrow} = f_0^s - f_0^a \tag{3.12}$$

For the ^{3}He quasiparticles in ^{3}He–^{4}He mixtures, the expressions (3.8) to (3.11), are likewise valid. The specific heat was measured by Anderson *et al.*$^{(2)}$ at $x = 1.3\%$ and 5%. The spin susceptibility χ_P has been measured by Anderson *et al.*$^{(2)}$ and Ahonen *et al.*$^{(20)}$ at several ^{3}He concentrations.

However, since one cannot measure directly the compressibility of the ^{3}He component in ^{3}He–^{4}He mixtures, one cannot obtain F_0^s from (3.8). It is possible, though, to obtain F_0^s from measurements of the second sound velocity. Khalatnikov(21) showed that at low concentration and zero temperature, the second sound velocity is

$$u_2^2(T=0)=\frac{V_F^2}{3}(1+F_0^s)\left(1+\frac{\tilde{F}_1^s}{3}\right)\left[1-x\frac{m_4}{m_i}\left(1+\alpha+\frac{m_i-m_3^0}{m_4}\right)^2\right] \tag{3.13}$$

where m_i, the "inertial mass' of ^{3}He, is related to the normal fluid density, $\rho_n=nm_i$, and $\tilde{F}_1^s$ is defined as

$$1+\frac{1}{3}\tilde{F}_1^s=\frac{m_3^*}{m_i} \tag{3.14}$$

if one assumes Galilean invariance. Corruccini(22) measured the second sound velocity u_2, and then calculated F_0^s by assuming $m_i=m^*(x=0)$ for all concentrations; he obtains $F_0^s=-0.11\pm 0.06$ at $x=1.3\%$ and $F_0^s=-0.26\pm 0.05$ at $x=5.0\%$. However, if we use the inertial mass, $m_i\simeq 2.0\,m_3^0$, at 5%, obtained by Sherlock and Edwards,(23) we find that $F_0^s\simeq -0.33$ at $x=5.0\%$. The results of F_0^s, F_0^a, and m^* obtained from experiment are summarized in Table I.

As may be seen in Table I, the present experimental uncertainty in F_0^s and F_0^a, or what is equivalent, f_0^s and f_0^a, as a function of concentration is such that one has a good deal of latitude in choosing these quantities. Moreover, we wish to determine the effective interactions over a wide range of concentrations and pressure for which experiment at present does not

Table I. Measured Quantities for ^{3}He–^{4}He Mixtures at $x=1.3\%$ and 5% (SVP)

x	1.3%	5%
m^*/m_3^0	2.38 ± 0.04[2]	2.46 ± 0.04[2]
F_0^s	-0.11 ± 0.06[22]	-0.26 ± 0.05[22]
F_0^a	0.09 ± 0.03[2]	0.08 ± 0.03[2]
		0.03 ± 0.02[20]
F_1^s	0.05 ± 0.05[a]	0.15 ± 0.05[a]
F_1^a		0.34 ± 0.1[9,2]
		0.18 ± 0.06[9,20]

[a] F_1^s is calculated from the "Galilean" relation: $m^*/m_3=1+1/3\,F_1^s$, where $m_3=2.34\,m_3^0$ is the ^{3}He effective mass as $x\to 0$.

provide a definitive guide to f_0^s. We therefore develop scaling laws to obtain $V_0^{\uparrow\uparrow}(x)$, $V_0^{\uparrow\downarrow}$, and $u_0(x)$; and hence $f_0^{\uparrow\uparrow}(x)$ and $f_0^{\uparrow\downarrow}(x)$. We consider the momentum-dependent interactions in the following sections.

In the zero concentration limit, $V_0^{\uparrow\downarrow}$ and u_0 are obtained by the thermodynamic arguments of BBP,

$$\lim_{x\to 0} V_0^{\uparrow\downarrow} = (1+2\alpha)\frac{m_4 s^2}{n_4} \tag{3.15}$$

$$\lim_{x\to 0} u_0 = (1+\alpha)\frac{m_4 s^2}{n_4} \tag{3.16}$$

where n_4 is the ^{4}He density and s is the sound velocity, while the long-wavelength limit of the density–density response function is given by Eq. (2.6). The limiting value of $V_q^{\uparrow\uparrow}$ can be obtained if one notes that the vanishing of the forward scattering amplitude for parallel spin quasiparticles in the zero concentration limit leads to the requirement that $f_0^{\uparrow\uparrow}(0)=0$. From Eqs. (2.6) (3.15), and (3.16), it follows that

$$\lim_{x\to 0} V_0^{\uparrow\uparrow} = (1+\alpha)^2\frac{m_4 s^2}{n_4} \tag{3.17}$$

$$\lim_{x\to 0} f_0^{\uparrow\downarrow} = -\alpha^2\frac{m_4 s^2}{n_4} \tag{3.18}$$

To scale these results to finite concentration, we first note that $V_0^{\uparrow\uparrow}$ in (3.17) is almost identical to the spatial average, $f_0^{\uparrow\uparrow}$, of the effective interaction between ^{3}He quasiparticles of parallel spin in pure ^{3}He at a pressure (~21 atm) such that its density is the same as ^{4}He at SVP. Indeed, as may be seen in Table II the difference of (3.17) and the $f_0^{\uparrow\uparrow}$ obtained by using the data of Greywall(24) is less than 1%. *We assume this result holds true at all concentrations*; thus, at a given ^{3}He concentration we first calculate the total particle density, then determine by extrapolation from the data of Greywall the value of $f_0^{\uparrow\uparrow}$ of pure ^{3}He at the same density, and finally get $V_0^{\uparrow\uparrow}\simeq f_0^{\uparrow\uparrow}$. The value of $V_0^{\uparrow\downarrow}$ at finite concentration is then fixed by the experimentally determined Landau parameter, $F_0^a=\frac{1}{2}N(0)(f_0^{\uparrow\uparrow}-f_0^{\uparrow\downarrow})$; we find that an acceptable fit to the latter can be obtained with the simple scaling law,

$$V_0^{\uparrow\downarrow} = (1+\beta x)(1+2\alpha)\frac{ms^2(x)}{n(x)} \tag{3.19}$$

where $\beta \cong 1.2$. The sound velocity of ^{4}He is found experimentally to be a

Table II. Comparison of the Interactions, $nf_0^{\uparrow\uparrow}$, $nf_0^{\uparrow\downarrow}$, $nV_0^{\uparrow\uparrow}$, $nV_0^{\uparrow\downarrow}$, and nu_0 (in Degrees Kelvin) in ^{3}He–^{4}He Mixtures at Various Concentrations; with the Corresponding Interactions $nf_0^{\uparrow\uparrow}$, $nf_0^{\uparrow\downarrow}$ in pure ^{3}He, Determined from the Experimental Results of Greywall,[24] and nf_0^s in Pure ^{4}He at the Same Densities

x (%)	0	1.3	2.4	5.0	6.0
n (Å^{-3})	0.02180	0.02172	0.02165	0.02149	0.02143
nf_0^s	27.3	26.7	26.2	25.2	24.7
nu_0	35.1	34.6	34.1	33.3	32.8
$nV_0^{\uparrow\downarrow}$	42.8	42.5	42.3	41.8	41.6
$nf_0^{\uparrow\downarrow}$(^{3}He)	46.5	45.8	45.2	43.8	43.3
$nV_0^{\uparrow\uparrow}$	45.0	44.3	43.7	42.4	41.9
$nf_0^{\uparrow\uparrow}$ (^{3}He)	45.2	44.5	42.8	42.4	41.9
$nf_0^{\uparrow\downarrow}$	-2.20	-2.14	-2.10	-2.01	-1.99
$nf_0^{\uparrow\uparrow}$	0	-0.34	-0.68	-1.45	-1.73

linear varying function of the particle density throughout the density range of interest for mixtures. The Landau parameter F_0^a is given by

$$F_0^a = \tfrac{1}{2}N(0)(f_0^{\uparrow\uparrow} - f_0^{\uparrow\downarrow}) = \tfrac{1}{2}N(0)(V_0^{\uparrow\uparrow} - V_0^{\uparrow\downarrow}) \tag{3.20}$$

since the phonon–roton induced interaction cancels. Our resulting values of F_0^a as a function of ^{3}He concentration are shown in Fig. 3, while the

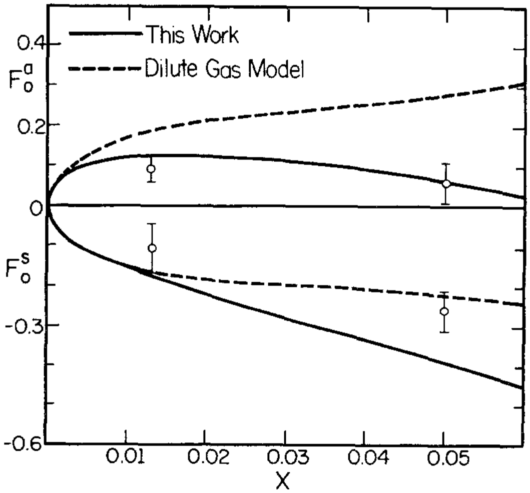


Fig. 3. The deduced values of F_0^s and F_0^a as functions of ^{3}He concentration. The experimental data are from Table I.

corresponding value of $V_0^{\uparrow\uparrow}$ and $V_0^{\uparrow\downarrow}$ are compared with those of pure ^{3}He at the same density in Table II.

To obtain the spatial average of the ^{3}He–^{4}He interaction at finite concentration, we assume that the BBP expression, Eq. (3.18), for the interaction between quasiparticles of opposite spin applies at finite concentration

$$f_0^{\uparrow\downarrow} = -\alpha^2 \frac{m_4 s^2(x)}{n(x)} \tag{3.21}$$

From (2.4a), $f_0^{\uparrow\downarrow}$ can be separated into direct and phonon-induced terms, according to

$$f_0^{\uparrow\downarrow} = V_0^{\uparrow\downarrow} + u_0^2 \chi(0, 0) \tag{3.22}$$

which, on making use of Eqs. (2.6) and (3.19), becomes

$$f_0^{\uparrow\downarrow} = (1 - \beta x)(1 + 2\alpha) \frac{m_4 s^2(x)}{n(x)} - u_0^2 \frac{n(x)}{m_4 s^2(x)} \tag{3.23}$$

From (3.21) and (3.23), we find that by assuming that u_0 increases with concentration according to

$$u_0 = (1 + \gamma x)(1 + \alpha) \frac{m_4 s^2(x)}{n(x)} \tag{3.24}$$

and taking $\gamma = 0.56$, we obtain the result (3.21). The corresponding values of u_0 are likewise given in Table II, while our deduced value of F_0^s is given in Fig. 3.

We see from Eqs. (3.19) and (3.24) that as the ^{3}He concentration increases (and the overall system density decreases) the spatial average of the direct interaction between antiparallel spin ^{3}He quasiparticles, when compared to that between ^{4}He particles, ms^2/n, increases, as does the effective ^{3}He–^{4}He interaction. The physical origin of this behavior is the enhanced role of the zero point motion of the ^{3}He atoms as the overall system density decreases; indeed we see that $\beta \cong 2\gamma$, consistent with our underlying physical picture of the low concentration limit, where zero point motion increases the ^{3}He–^{4}He interaction by factor of $(1 + \alpha)$ and the effective interaction between antiparallel spin ^{3}He quasiparticles by a factor of $(1 + 2\alpha)$.

On comparing the behavior of $V_0^{\uparrow\uparrow}$ with that of $V_0^{\uparrow\downarrow}(x)$ we see the combined influence of the Pauli principle and zero point motion as a function of concentration. The first point to be noticed is that as in the case for pure ^{3}He the statistical correlations arising from the Pauli principle represent a comparatively small correlation to those arising from the combination of the strong short-range particle interaction and the zero point motion of the ^{3}He quasiparticles. Thus, in the very dilute limit, the spatial average of the direct effective interaction between parallel spin ^{3}He quasiparticles is only $(\alpha^2 ms^2/n)$ greater than that between antiparallel spin quasiparticles. This effect may be attributed to a Pauli principle enhancement in dilute mixtures of the way in which zero point motion increases the effective ^{3}He quasiparticle interaction over that of the background ^{4}He quasiparticles. This "Pauli enhancement" decreases with increasing ^{3}He concentration [it is only $\sim 0.3\alpha^2(ms^2/n)$ at $x = 0.05$], a result which reflects the fact that the influence of the Pauli principle is largest in the low-density limit.

It is also instructive to compare the effective direct interaction between antiparallel spin ^{3}He atoms in dilute ^{3}He–^{4}He mixtures with those in pure ^{3}He at the same density. As may be seen in Table II, in the very low concentration limit $V_0^{\uparrow\downarrow} \simeq f_0^{\uparrow\downarrow}(^3\mathrm{He}) - 1.7\,\alpha^2 ms^2/n$, while at 5% concentrarion we have $V_0^{\uparrow\downarrow} \simeq f_0^{\uparrow\downarrow}(^3\mathrm{He}) - \alpha^2 ms^2/n$. It would seem that for a given pair of antiparallel spin ^{3}He quasiparticles in pure ^{3}He the net result of replacing all the intervening ^{3}He atoms (of either spin) by ^{4}He atoms, while keeping the total density fixed, is to decrease the average effective direct repulsion between the atoms by an amount $\sim\alpha^2 ms^2/n$; put another way, in the very dilute mixture, the strength and/or the range of the short-range repulsion is reduced compared to pure ^{3}He. [An equivalent statement is that in the mixture zero point motion is somewhat less effective in increasing the strength of the net repulsion between ^{3}He quasiparticles of antiparallel spin (compared to ^{4}He at the same density) than it is for pure ^{3}He.]

We have remarked that the spatial average, $V_0^{\uparrow\uparrow}$, in the very dilute mixture is within 1% of that for a pair of ^{3}He quasiparticles in pure ^{3}He at the same density, and have proposed that this same situation obtains at finite concentrations. This situation likely comes about as a result of two opposing effects: as with antiparallel spin particles, replacing the ^{3}He atoms by ^{4}He atoms decreases the average effective direct repulsion by an amount of magnitude $\sim\alpha^2(ms^2/n)$; on the other hand, Pauli principle correlations act to increase this repulsion by an amount of the same magnitude.

In concluding this discussion we see in Table II that in the mixtures both $f_0^{\uparrow\downarrow}$ and $f_0^{\uparrow\uparrow}$ remain of order $\alpha^2 ms^2/n$ as the ^{3}He concentration increases [the first, $f_0^{\uparrow\downarrow}$, by virtue of our assumption, Eq. (3.21)]. We note that $f_0^{\uparrow\uparrow}$ becomes increasing attractive as the concentration increases, and at the highest concentrations begins to be comparable to $f_0^{\uparrow\downarrow}$; thus as the ^{3}He concentration increases the Pauli principle bcomes less effective in reducing the net interaction between ^{3}He quasiparticles, and the overall tendency for this interaction to be attractive wins out.

3.4. $u(r, x)$, $V^{\uparrow\downarrow}(r, x)$, and $V^{\uparrow\uparrow}(r, x)$

Given the spatial averages of the pseudopotentials, $u(r)$, $V^{\uparrow\downarrow}(r)$, and $V^{\uparrow\uparrow}(r)$, determined as a function of concentration by the scaling laws described above, it follows that for a given shape, each interaction may be characterized by either its core height $(a_{34}, a^{\uparrow\downarrow}, a^{\uparrow\uparrow})$ at the origin or the range $(r_{34}, r^{\uparrow\downarrow}, r^{\uparrow\uparrow})$ of its soft-core repulsion. In contrasting each of these effective interactions with that between two background ^{4}He atoms in the liquid, at any given density there are two "extreme" possibilities:

(i) The very-short-range part of the configuration space interaction is essentially unchanged when one replaces a ^{4}He atom by a ^{3}He quasiparticle—in which case the entire burden of the change from $nf^s_{0,4} = ms^2$ is born by an increase in the range of the soft-core repulsion compared to 2.68 Å.

(ii) The range of the short-range repulsion is essentially unchanged; the entire burden of the increase in strength of the various spatially averaged direct interactions is then born by an increase in the net strength of the short-range repulsion interaction.

To be more specific, let us compare the ^{3}He–^{4}He quasiparticle interation, in the very dilute limit, with $f^s_4(r)$, the effective interaction between ^{4}He atoms at the same density.

As we have seen, when one replaces a single ^{4}He atom by a ^{3}He atom, the enhanced zero point motion of the latter gives rise to a spatially averaged effective ^{3}He–^{4}He particle interaction which is $\sim\alpha(\sim 30\%)$ larger than that between two ^{4}He atoms. On the first alternative (an increase in r_{34}), this increase comes about entirely at the expense of the attractive part of the effective interaction; it is associated with a change in the effective interaction at distances r such that $2.68\ \text{Å} \lesssim r \lesssim r_{34} \cong 2.75\ \text{Å}$. On the second alternative, $r_{34} \cong 2.68$ Å, and the $\sim$30% increase in the spatially averaged effective interaction is associated with changes in the effective interaction at distances small compared to 2.68 Å. Given a choice of one extreme or the other, the first is far more appealing on physical grounds, since the substantially enhanced (compared to a ^{4}He atom) zero point motion of a ^{3}He atom strongly suggests the need for modification in the configuration space interaction at distances $\gtrsim$2.68 Å. In what follows therefore we pursue the consequences of this alternative for both $u(r)$ and $V^{\uparrow\downarrow}(r)$. We have in fact, used pseudopotentials based on the latter assumption ($r_c \equiv 2.68$ Å) to carry out calculations of transport properties and mode–mode coupling with results which are in far less satisfactory agreement with experiment than those using the first alternative. We have also examined elsewhere$^{(15)}$ the consequences of modifications in the effective interaction associated with some changes in short-range ($r \lesssim 2.68$ Å) part of the interaction, without, however, embracing the unphysical extreme of attributing all physical effects to such short-range modifications. Such an approach involves introducing an additional free parameter, and does not yield better results for either transport properties or mode–mode coupling than the minimal assumption [no changes in $f^{\uparrow\downarrow}(r)$ or $u(r)$ from $f^s(r)$ for $r \lesssim 2$ Å] we now explore.

We thus take all pseudopotentials assumed to be of the AP form, Eq. (3.1); a_{34}, the core height for $u(r)$, and $a^{\uparrow\downarrow}$, the core height for $V^{\uparrow\downarrow}(r)$, are taken to be the same as the core height of ^{4}He–^{4}He potential, a, at the same density

$$a^{\uparrow\downarrow} \equiv a_{34} \equiv a \tag{3.25}$$

From (3.15) and (3.16) it follows that the effective range, r_{34}, of the repulsive interaction between ^{3}He and ^{4}He atoms must be greater than 2.68 Å, while that between ^{3}He atoms of antiparallel spin, $r^{\uparrow\downarrow}$, is greater yet.

In considering the role played by the Pauli principle in determining the form of $V^{\uparrow\uparrow}(r)$, we recall that for $x \lesssim 0.06$, $V_0^{\uparrow\uparrow} > V_0^{\uparrow\downarrow}$ (f_0^a is positive), so that for these concentrations Pauli principle correlations, which keep, on average, particles of parallel spin farther apart than those of antiparallel spin, enhance the role played by zero point oscillations in increasing the strength of the direct interaction between parallel spin particles over that of ^{4}He quasiparticles. Were one to choose $a^{\uparrow\uparrow} = a^{\uparrow\downarrow}$, it follows that $r^{\uparrow\uparrow} > r^{\uparrow\downarrow}$. We introduce the parameter δ, defined in Eq. (3.4), to characterize the interaction between parallel spin quasiparticles in this model. Since Pauli principle correlations act at distrances <2.68 Å as well, we further expect that the screening of the short-range part of the effective interaction between ^{3}He atoms of parallel spin will be somewhat more effective than that assumed for antiparallel spin atoms, so that one should have

$$a^{\uparrow\uparrow} \lesssim a^{\uparrow\downarrow} \tag{3.26}$$

We deal with the consequence of Eq. (3.26) by regarding δ as a free parameter of the model, one which can be varied as a function of concentration to obtain the "best" subsequent scattering amplitudes, as measured by the agreement between theory and experiment for the transport coefficients. Physically, as is the case for pure ^{3}He, δ gives a measure of the effectiveness of the Pauli principle. We expect on physical grounds that this "Pauli principle parameter," δ, will be larger for solutions with a smaller ^{3}He concentration. This turns out to be the case in practice; our best fit (from transport calculations) values of δ turn out to be $\delta \cong 0.13$ Å for $x = 1.3\%$ and $\delta \cong 0.03$ Å for $x = 5\%$. The resulting potentials are summarized in Table III.

It is interesting to compare the core height, $a^{\uparrow\downarrow}$, of the direct interaction

Table III. Comparison of the Core Heights and the Core Radius of the Pseudopotentials: $u(r)$, $V^{\uparrow\uparrow}(r)$, $V^{\uparrow\downarrow}(r)$, in ^{3}He–^{4}He Mixtures and $f^{\uparrow\uparrow}(r)$, $f^{\uparrow\downarrow}(r)$, in Pure ^{3}He; Core Heights, a_{34}, $a^{\uparrow\uparrow}$, and $a^{\uparrow\downarrow}$ in Degrees Kelvin; Core Radii, r_{34}, $r^{\uparrow\uparrow}$, and $r^{\uparrow\downarrow}$, are in Å

System	X (%)	P (atm)	a_{34}	r_{34}	$a^{\uparrow\downarrow}$	$r^{\uparrow\downarrow}$	$a^{\uparrow\uparrow}$	$r^{\uparrow\uparrow}$
^{3}He–^{4}He	0	0	49.3	2.751	49.3	2.821	39.8	2.993
^{3}He		20.6			55.7	2.773	54.2	2.780
^{3}He–^{4}He	1.3	0	48.9	2.752	48.9	2.825	41.6	2.956
^{3}He		20.1			54.8	2.779	53.3	2.785
^{3}He–^{4}He	2.4	0	48.6	2.753	48.6	2.829	42.8	2.931
^{3}He		19.6			54.0	2.785	52.5	2.792
^{3}He–^{4}He	5.0	0	47.9	2.756	47.9	2.827	52.5	2.870
^{3}He		18.5			52.3	2.796	50.9	2.803
^{3}He–^{4}He	6.0	0	47.7	2.757	47.7	2.840	47.4	2.846
^{3}He		18.2			51.7	2.800	50.3	2.807

between two spin-antiparallel ^{3}He quasiparticles in ^{3}He–^{4}He mixtures with that in pure ^{3}He. We see that in the small concentration limit, $a^{\uparrow\downarrow}$ (pure ^{3}He) $> a^{\uparrow\downarrow}(^3\mathrm{He}\text{–}^4\mathrm{He})$. The physical origin of this result is the same as that which leads one to conclude that $a^{\uparrow\downarrow}$ (pure ^{3}He) $> a$ (pure ^{4}He); because of the larger zero point motion of ^{3}He atoms, the screening of the short-range hard core is somewhat less effective than that found for ^{4}He atoms. We expect the consequences of zero point motion will be larger for pure ^{3}He and smaller when we replace all but two ^{3}He atoms with ^{4}He atoms. As the total density decreases, the consequences of zero point motion will be weaker, hence the difference between the core heights $a^{\uparrow\downarrow}$ (pure ^{3}He) and $a^{\uparrow\downarrow}$ (^{3}He–^{4}He) is smaller. A similar argument can be applied to $a^{\uparrow\uparrow}$, as one can see from Table III. The difference $a^{\uparrow\uparrow}(^3\mathrm{He}) - a^{\uparrow\uparrow}(^3\mathrm{He}\text{–}^4\mathrm{He})$ is larger than $a^{\uparrow\downarrow}(^3\mathrm{Hz}) - a^{\uparrow\downarrow}(^3\mathrm{He}\text{–}^4\mathrm{He})$ at very low concentrations, because in this limit the Pauli principle is more effective for dilute ^{3}He–^{4}He solutions than for pure ^{3}He. The Pauli principle further reduces the possibility that ^{3}He atoms "feel" the short-distance repulsion, and hence $a^{\uparrow\uparrow}$ *in* ^{3}He–^{4}He mixtures is pushed further down.

The pseudopotentials $V^{\uparrow\uparrow}(r)$, $V^{\uparrow\downarrow}(r)$ and $u(r)$ for ^{3}He–^{4}He mixtures and the ^{4}He–^{4}He interaction, $f_4^s(r)$, at the same density are shown in Fig. 4 for $x = 5\%$, while their Fourier transforms, $V_q^{\uparrow\uparrow}$, $V_q^{\uparrow\downarrow}$, u_q, and $f_{q,4}^s$, are shown in Fig. 5. We compare, in Fig. 6, these interactions in the $x \to 0$ limit with the BHP(18) effective interaction between ^{3}He quasiparticles in pure liquid ^{3}He at the same density. Note that while there are significant differences between the various interactions at low momentum transfer ($q \lesssim 0.8$ Å^{-1}); for

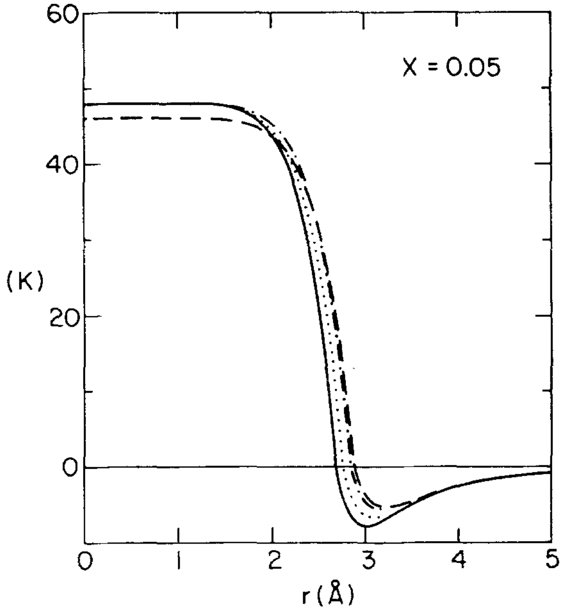


Fig. 4. Model pseudopotentials for $x = 5\%$: $V^{\uparrow\uparrow}(r)$ (dashed curve), $V^{\uparrow\downarrow}(r)$ (dash–dotted curve), $u(r)$ (dotted curve), and $f_4^s(r)$ (solid curve).

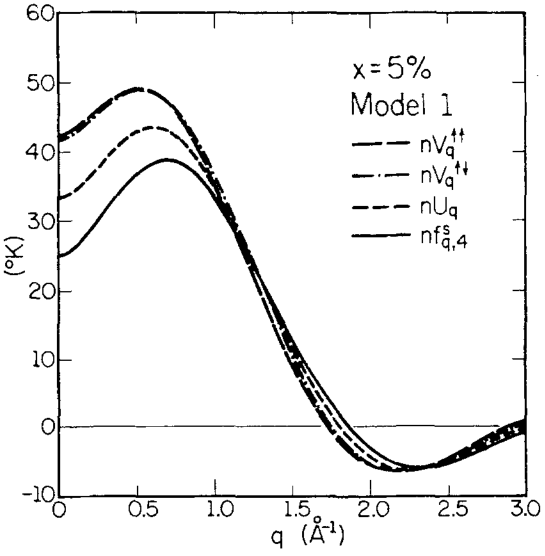


Fig. 5. The momentum-dependent pseudopotentials $f^s_{q,4}$, $V_q^{\uparrow\uparrow}$, $V_q^{\uparrow\downarrow}$, and u_q for $x = 5\%$.

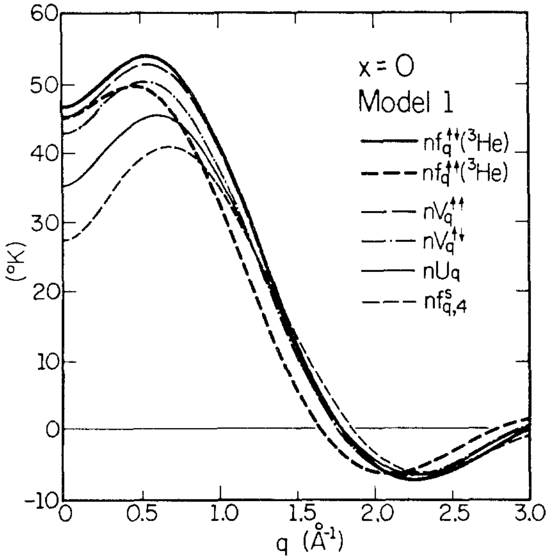


Fig. 6. The pseudopotentials $f_{q,4}^s$, $V_q^{\uparrow\uparrow}$, $V_q^{\uparrow\downarrow}$ and u_q in the dilute limit, and the corresponding potentials $f_q^{\uparrow\uparrow}$ and $f_q^{\uparrow\downarrow}$ for pure ^{3}He at the same density.

$q \geqslant 1.5$ Å^{-1}, the momentum dependence and magnitude of the interactions display a striking similarity.

3.5. $f_q^{\uparrow\downarrow}$ and $f_q^{\uparrow\uparrow}$

Given the results for the ^{3}He–^{4}He quasiparticle interaction and the direct part of the ^{3}He–^{3}He quasiparticle interaction, the remaining quantity which is needed to specify the momentum-dependent effective interactions is the static density–density response function, $\chi(q, 0)$, which AP express in the form, Eq. (2.13). In the very dilute limit ($x \to 0$) we can determine $\chi(q, 0)$ by using the AP fit to the results of Cowley and Woods$^{(25)}$; the AP results for $-1/\chi(q, 0)$, $f_{q,4}^s$, and $-1/\chi_{44,\text{sc}}(q, 0)$ are shown in Fig. 7. At finite concentration, we calculate $\chi(q, 0)$ by scaling the AP results to lower densities. Our resulting effective potentials are shown in Fig. 8. We comment briefly.

We note first that at all concentrations $f_q^{\uparrow\downarrow}$ has the same general shape as the phenomenological BBP interaction; our work thus serves to justify the general character of the BBP interaction although, as may be seen, our interaction becomes repulsive at a somewhat lower momentum transfers than that proposed by BBP. As may be seen from Eq. (2.13) and Fig. 7, the

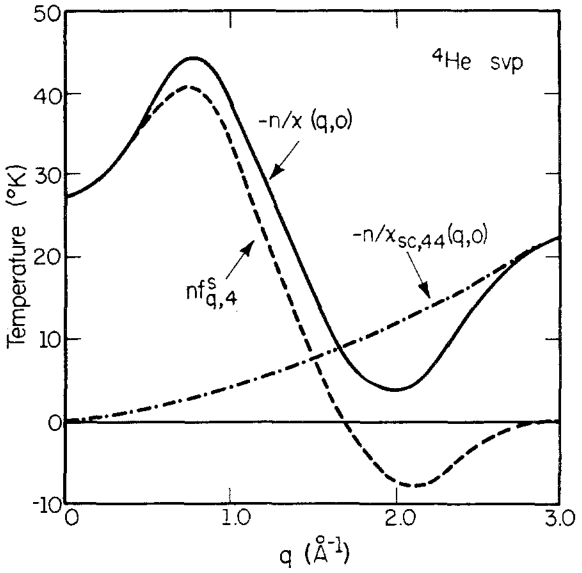


Fig. 7. The quantities $-1/\chi(q, 0)$, $f^s_{q,4}$ and $-1/\chi_{44,sc}(q, 0)$ for liquid ^{4}He at SVP.

physical reason for the change in sign of the effective interaction may be traced to the fact that $\chi(q, 0)$ falls off more rapidly with increasing momentum transfer than the direct interactions increase; were $\chi(q, 0)$ to be comparatively flat for low q, $f_q^{\uparrow\downarrow}$ would actually become more attractive with increasing q (since u_q^2 increases more rapidly with increasing q than does $V_q^{\uparrow\downarrow}$). The concentration dependence of $f_q^{\uparrow\downarrow}$ is seen to be comparatively slight.

Our results for $f_q^{\uparrow\uparrow}$ display a more striking concentration dependence, which reflects the concentration dependence of the Pauli principle correlations in dilute mixtures. In the very low concentration limit we see that this interaction (here determined for the first time) is essentially repulsive; the slight additional spin-induced repulsion $(\sim\alpha^2 ms^2/n)$ is sufficient to bring this about. On the other hand, by the time one has reached a 5% concentration, where, as we have seen, the Pauli principle correlations play a far smaller role, $f_q^{\uparrow\uparrow}$ displays qualitatively the same behavior as $f_q^{\uparrow\downarrow}$.

The change in sign of the effective interactions at finite momentum transfers is the physical origin of the exceedingly low temperatures we calculate for the ^{3}He superfluid transition in ^{3}He–^{4}He mixtures. As the concentration increases, one samples an increasingly large fraction of the repulsive part of the interaction between antiparallel spin particles—hence the calculated decrease of T_c with x. We see also in the next section that our

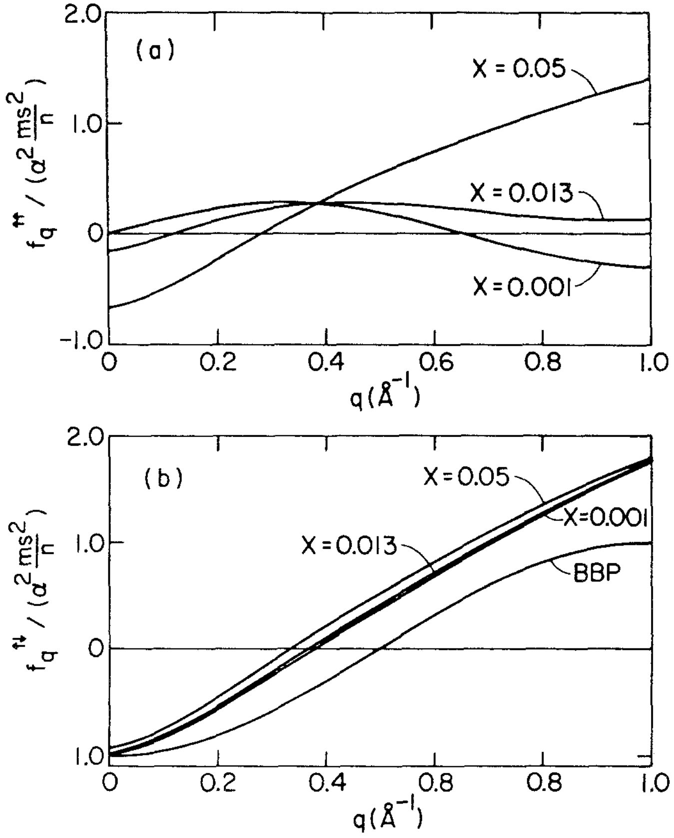


Fig. 8. (a) The effective interaction $f_q^{\uparrow\uparrow}$ of ^{3}He quasiparticles for concentrations $x = 0.001$, $x = 0.013$, and $x = 0.05$. (b) The effective interaction $f_q^{\uparrow\downarrow}$ of ^{3}He quasiparticles. Also shown is the BBP potential, Eq. (1.1).

calculated Pauli-principle-induced changes in $f_q^{\uparrow\uparrow}$ give rise in turn to maxima in transpot coefficients at ~4% concentrations.

3.6. Pseudopotentials at Higher Pressures

We have extended our calculations of the concentration dependence of the effective interactions in ^{3}He–^{4}He mixtures to finite pressures, with the results shown in Figs. 9–11. We comment on our calculations briefly.

1. As is the case for SVP, we assume $V_0^{\uparrow\uparrow} = f_0^{\uparrow\uparrow}$ (pure ^{3}He) at all concentrations (and pressures).

2. $V_0^{\uparrow\uparrow}$ is readily determined once χ_p (or what is equivalent, F_0^a) is

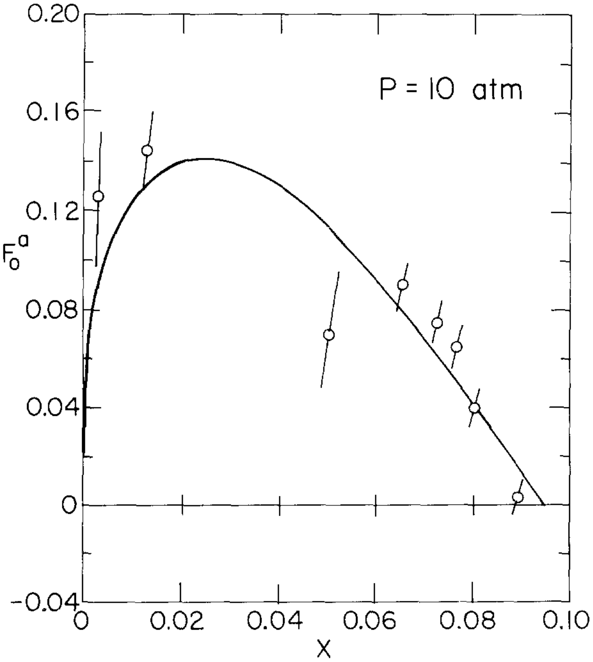


Fig. 9. A comparison of our "scaling law" results for F_0^a with experiment at 10 atm. The experimental results are obtained by combining the results of Ahonen *et al.*(20) with the effective mass results of Greywall.(31)

known. We find that, as is the case for SVP, a good fit to the measured values of $F_0^a(x)$ can be obtained with an expression of the form given by Eq. (3.19), where the quantities α and β are compared with the corresponding values at SVP in Table IV. Our scaling law concentration dependence of F_0^a at pressures of 10 atm and 20 atm is compared with experiment in Figs. 9 and 10. Note that both α and β decrease with increasing pressure; both are a consequence of the zero point oscillations of the ^{3}He atoms, and the relative importance of the latter decreases with increasing pressure.

3. As is the case for SVP, we assume that $f_0^{\uparrow\downarrow}$, the net interaction between antiparallel spin ^{3}He atoms *in the mixture*, is given by Eq. (3.21). It follows that u_0, the spatial average of the ^{3}He–^{4}He effective interaction, may still be written in the form, Eq. (3.24). Our calculated values of γ at 10 and 20 atm are compared with those in svp in Table I; note that at these pressures, we continue to have $\gamma \simeq \beta/2$.

4. The effective interactions, $V_0^{\uparrow\uparrow} \cong f_0^{\uparrow\uparrow}$, and $V_0^{\uparrow\downarrow}$, as a function of density are compared with $f_0^{\uparrow\downarrow}$ for pure ^{3}He in Fig. 11. The presence of the

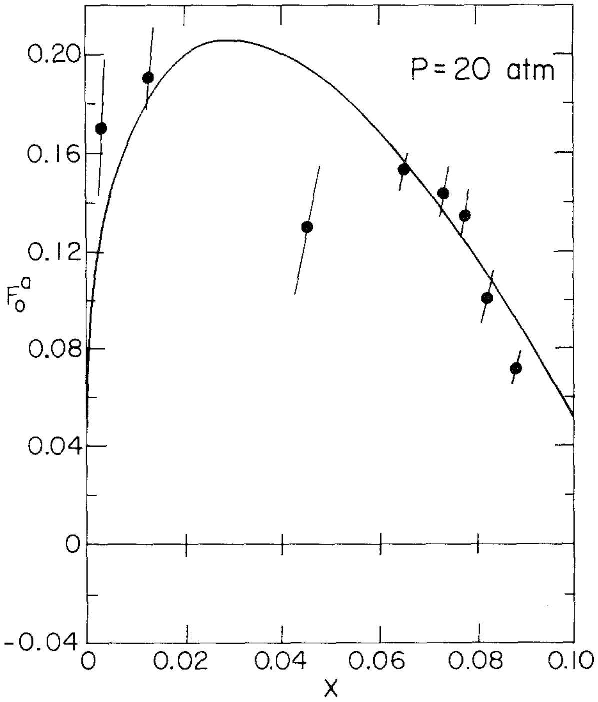


Fig. 10. A comparison of our "scaling law" results for F_0^a with experiment at 20 atm. The experimental results are obtained by combining the results of Ahonen *et al.*[20] with the specific heat results of Greywall.[31]

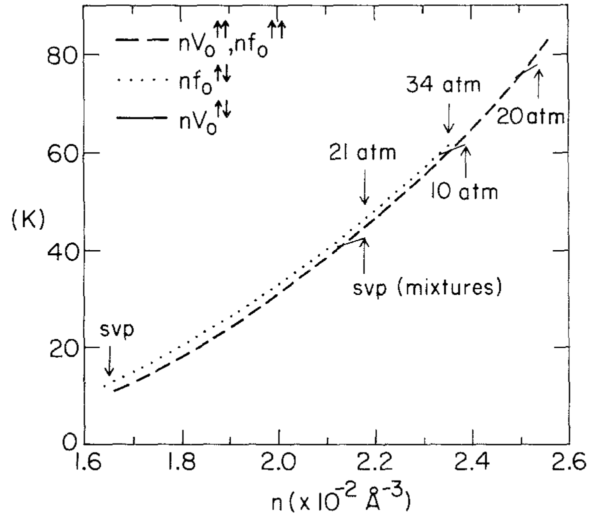


Fig. 11. A comparison of the spatial averages, $V_0^{\uparrow\downarrow}$ and $V_0^{\uparrow\uparrow}$, of the pseudopotentials, $V^{\uparrow\downarrow}(r)$ and $V^{\uparrow\uparrow}(r)$, which describe the direct interactions between ^{3}He atoms in dilute ^{3}He–^{4}He mixtures, with their counterparts, $f_0^{\uparrow\downarrow}$ and $f_0^{\uparrow\uparrow}(\cong V_0^{\uparrow\uparrow})$, for pure ^{3}He. The corresponding experimental pressures for both pure ^{3}He and ^{3}He–^{4}He mixtures are indicated by arrows.

Table IV. The Excess Molar Volume, Dilute Limit Effective Mass, and Scaling Parameters as a Function of Pressure

P	α	β	γ	m_3/m_0
SVP	0.284	1.2	0.56	2.34
10	0.208	0.6	0.29	2.57
20	0.165	0.47	0.23	2.85

^{4}He atoms in the mixtures enables one to reach (at 20 atm) a higher system density than can be attained with pure ^{3}He; to obtain $V_0^{\uparrow\uparrow}$ at these densities we have extrapolated from lower densities in the manner shown. One sees that in the mixtures the Pauli principle "enhancement" of the interaction between parallel spin ^{3}He quasiparticles (which is largest at low concentrations) persists at higher pressures, and that at a given pressure, near the maximum observable concentration of ^{3}He atoms, one has $V_0^{\uparrow\uparrow} \cong V_0^{\uparrow\downarrow}$. Note that the differences between the spatial averages of the various interaction depicted there are $\lesssim \alpha^2 m_4 s^2(x)/n_4$.

5. These spatial averages are then used to construct the configuration space pseudopotentials, $u(r, x)$, $V^{\uparrow\downarrow}(r, x)$, and $V^{\uparrow\uparrow}(r, x)$ at 10 and 20 atm in a fashion which is identical to that employed at SVP. Thus the core height of both $u(r)$ and $V^{\uparrow\downarrow}(r)$ are taken to be the same as that found for $f^s(r)$ at the corresponding density, while the core height of $V^{\uparrow\uparrow}(r)$ is determined by our choice of δ. For each concentration, the "Pauli principle" parameter $\delta \equiv r^{\uparrow\uparrow} - r^{\uparrow\downarrow}$ is assumed to be pressure independent. At SVP, we found that $\delta = 0.13$ Å at $x = 1.3\,\%$ and $\delta = 0.03$ Å at $x = 5\,\%$. For $x = 8.5\,\%$, we chose $\delta \cong 0.01$ Å, which is close to the value $\delta = 0.007$ Å found by Bedell and Pines for pure ^{3}He at pressures $\geqslant 21$ atm).

4. TRANSPORT COEFFICIENTS AND SUPERFLUID TRANSITION TEMPERATURE

One test of the correctness of the pseudopotentials determined in the previous section is whether, as is the case for the analogous pseudopotentials for pure ^{3}He, one can use these to obtain the scattering amplitudes for ^{3}He quasiparticles to a good degree of accuracy. For a given set of scattering amplitudes it is relatively straightforward to calculate the transport properties (thermal conductivity, spin diffusion, viscosity) and obtain an upper limit for the transition of the ^{3}He quasiparticles to a superfluid state. The test then of the accuracy of the scattering amplitudes is whether these provide a good account of the experimentally determined transport properties and limits on the superfluid transition temperature.

We here report briefly on our results, which have been obtained in collaboration with Kevin Bedell, for the transport properties and superfluid transition temperatures as a function of concentration and pressure. A preliminary account of our calculations at SVP is given in Hsu,(15) and a detailed account of our work will be published elsewhere.(26)

4.1. Transport Coefficients at SVP

The procedure we follow to calculate the transport coefficients is a follows:

(1) For each concentration, a trial pseudopotential, $V_q^{\uparrow\uparrow}$, is determined as a function of the single free parameter δ, [Eq. (3.4)].

(2) The effective potentials $f_q^{\uparrow\uparrow}$ and $f_q^{\uparrow\downarrow}$ are then calculated by Eq. (2.4).

(3) The Bethe–Salpeter equation is used to construct scattering amplitudes from these effective potentials. Because the latter have not been derived from a microscopic vortex function, they do not automatically possess the right symmetry properties; correctly symmetrized amplitudes are obtained from the nonsymmetric results by using the Bedell–Pines ansatz.(19)

(4) In solving the Bethe–Salpeter equation it is assumed that the particle-hole effective masses which enter are momentum independent; it is further assumed that

$$F_1^a(x) \simeq F_1^s(x) = 4.14x - 22.86x^2 \qquad (4.1)$$

these values of F_1^s and F_1^a are consistent with the experimental results of Anderson *et al.*(2) and Owers-Bradley *et al.*(9)

(5) The thermal conductivity, κ, spin diffusion coefficient, D, and viscosity η, are then calculated by inserting the appropriate angular averages of the symmetrized scattering amplitudes into the exact solutions of the linearized Landau kinetic equations.(27)

(6) The above procedure is repeated for a variety of trial pseudopotentials (determined by our choice of δ) until we obtain scattering amplitudes which yield good agreement with experiment for D at $x = 1.3\%$ and 5%. Our results at SVP are summarized in Table V. As may be seen there it is possible to obtain very satisfactory agreement with experiment for κ and D, but the agreement is less satisfactory for η.

A similar problem with calculations of η exists for the theories of Ebner(5) and Bashkin(8); on the other hand, while the agreement between the theory of Fu and Pethick(6) and experiment is good for η, their results for D are not in agreement with the more recent experimental determinations.(7)

Table V. Comparison of Experimental and Various Theoretical Values of Transport Coefficients in ^{3}He—^{4}He Mixtures; units: κT[erg/sec cm], DT^2[(cm-m K)^2/sec], ηT^2[μ P-K^2]

	$x = 1.3\%$			$x = 5\%$		
	κT	DT^2	ηT^2	κT	DT^2	ηT^2
Exp.	11 ± 1.1[4]	18 ± 3[7]	0.034 ± 0.003[29]	24 ± 2.4[4]	75 ± 8[7]	0.28 ± 0.02[29]
Exp.		17.2 ± 1.7[2]			90 ± 9[2]	
Hsu and Pines	11.4	17.2	0.054	24.2	74.8	0.42
BBP	15	17	—	52	90	—
Ebner	9.6	18.6	—	27	80	—
Fu and Pethick	10.6	18.3	0.0322	24.9	87.2	0.284
Bashkin	14	22	0.05	34	65	0.38

We have tried a number of different parameters β and γ in Eq. (3.19) and Eq. (3.24) to reduce the values of viscosity, and find that if the viscosity is reduced by 50%, the thermal conductivity is also reduced by about the same percentage. Fisk and Hall(28) have pointed out that the measured value of η may be increased by 50% if the slip correction is included in the extrapolation of experimental data. It would seem that further investigations, both theoretical and experimental, are needed to understand this inconsistency between theory and experiment.

We have calculated the transport coefficients for all concentrations, $0 \leqslant x \leqslant 6.5\%$, by using a simple scaling law to determine δ: $\delta(x) = (0.16 - 2.5x)$ Å; this scaling law gives the "best fit" values of δ at $x = 1.3\%$ and 5%, and enables us to extrapolate those results to other concentrations. Our results for the thermal conductivity, κ; spin diffusion D, and viscosity η, as functions of concentration are shown in Figs. 12–14. We note that the spin diffusion coefficient, D, possesses a maximum at $x \sim 4\%$, and decreases with concentration for large x, in agreement with experiment. The thermal conductivity also shows this behavior. The results of the dilute gas model, in which the interaction between quasiparticles is described by a single parameter, a, the s-wave scattering length, are also shown. We note that for $x \lesssim 0.5\%$, the dilute gas model yields results comparable with ours. For larger x it fails to give overall quantitative agreement with experiments. This result leads us to conclude that for $x \gtrsim 1\%$, the momentum and concentration dependence of the effective interaction plays a significant role.

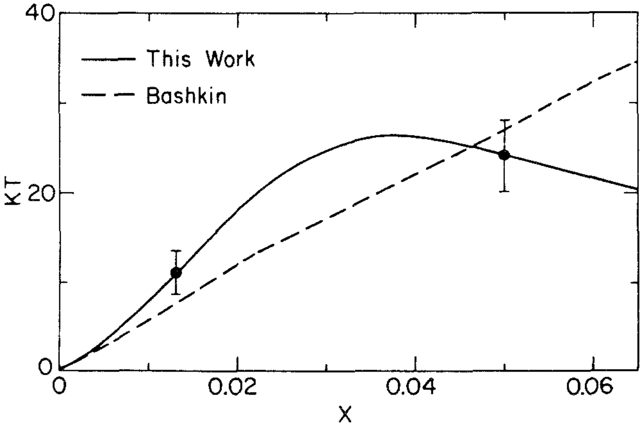


Fig. 12. The thermal conductivity coefficient, κ, as a function of concentration x. κT is in units of [erg/sec cm]. The experimental results are from Ref. 4.

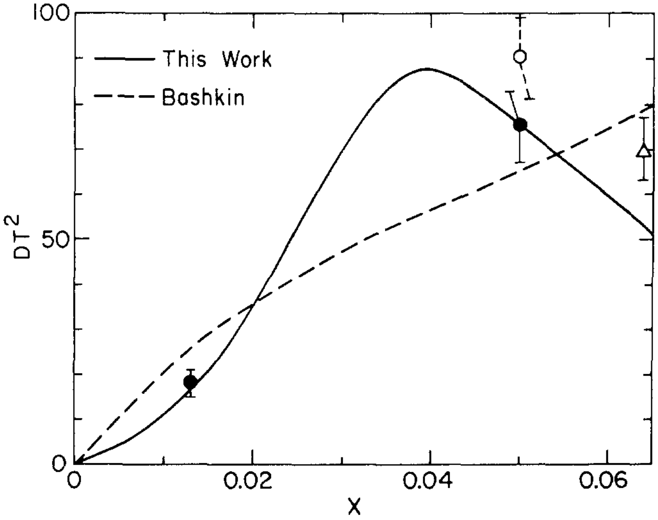


Fig. 13. The spin diffusion coefficient, D, as a function of concentration x. DT^2 is in units of [cm-(m K)^2/sec]. The experimental results are from Ref. 2 (dot), Ref. 7 (circle) and Ref. 30 (triangle).

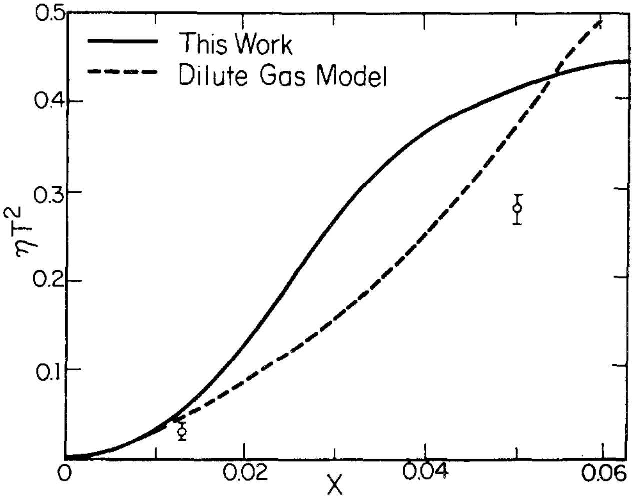


Fig. 14. The viscosity coefficient, η, as a function of concentration x. ηT^2 is in units of [μP K^2]. The experimental results are from Ref. 30.

4.2. Superfluid Transition Temperature

The normal-superfluid transition temperature, T_c, is given by BCS theory$^{(29)}$ as

$$T_c^l = 1.14\, a_l T_F \exp(-1/|g_l|) \tag{2.57}$$

where l is the angular momentum for the ^{3}He Cooper pair, a_l is a cutoff parameter, which takes into account the possible frequency dependence of the effective interactions, and g_l is given by

$$g_l = \frac{N(0)}{8} \int_{-1}^{1} d(\cos\phi)\, t^l(\pi, \phi)\, P_l(\cos\phi) \tag{2.58}$$

where $l = O(1)$ for singlet (triplet) state.

We have calculated T_c^l using our "best fit" scattering amplitudes. Our results are shown in Fig. 15; we have taken $a_l = 1$, so that our results represent an upper bound for T_c. It is interesting to see that for $x \lesssim 2.5\%$ s-state pairing is favored, while for $x \gtrsim 2.5\%$ p-state pairing is preferred. The magnitude, however, of T_c is of the order $\lesssim 10^{-8}$ K, far below the lowest temperature currently accessible. The maximum T_c occurs at $x \cong 0.75\%$. The existence of this maximum can be understood as follows: as x increases, so does the density of states $N(0) \sim x^{1/3}$; however, ^{3}He quasiparticles with $q \sim 2q_F$ increasingly sample the less attractive region of the effective interaction, $f_q^{\uparrow\downarrow}$ (for low concentration, we can nglect $f_q^{\uparrow\uparrow}$); the combination of the two effects give the resulting maximum in T_c.

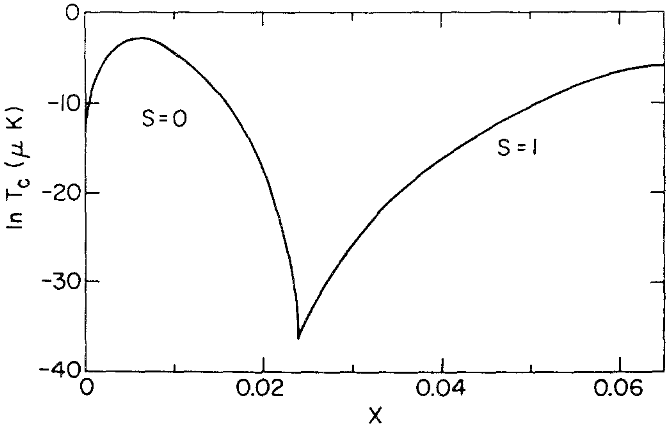


Fig. 15. The calculated normal-superfluid transition temperature, T_c, of ^{3}He quasiparticles as a function of concentration x.

4.3. Transport Coefficients at Elevated Pressures

We have calculated the transport coefficients as a function of concentration at pressures of 10 and 20 atm, with the results shown in Figs. 16–19. In carrying out these calculations we have followed the same procedures as described above for SVP mixtures, with the following exceptions:

(i) The effective mass as $x \to 0$, m_3 is obtained from Sherlock and Edwards,(23) while the finite concentration effective mass, $m^*/m_3 = 1 + \frac{1}{3}F_1^s$, is parametrized to give the best spin diffusion result compared with experiments. We found that the "best" $F_1^s(x)$ can be parametrized as follows:

$$F_1^s(x) = 7.5x - 60x^2 \qquad (10\text{ atm})$$

$$F_1^s(x) = 7.8x - 63x^2 \qquad (20\text{ atm})$$

Note that with these values of $F_1^s(x)$, we find a maximum in the effective mass as a function of concentration at $x \simeq 6\%$.

(ii) The behavior of D at $x \gtrsim 5\%$ (see Figs. 16 and 17) depends on both $F_1^s(x)$ and $\delta(x)$. Good agreement with experiment can also be achieved for different values of F_1^s if a slightly different value of δ is used.

The agreement between theory and experiment for the spin diffusion coefficient is seen to be excellent. Measurements of κ and η at finite pressure will be of great interest.

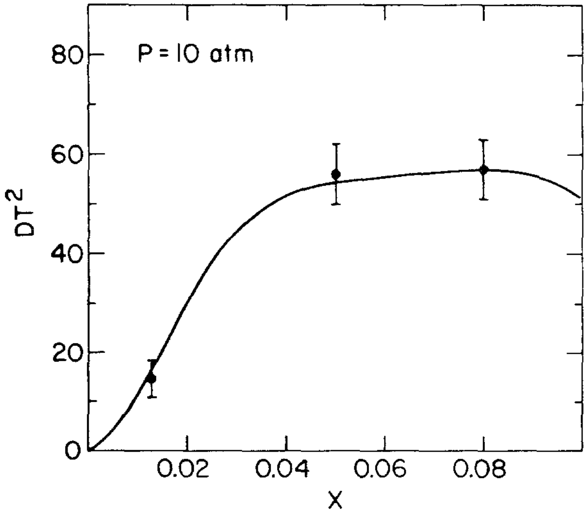


Fig. 16. Comparison of theory and experiment(7) for spin diffusion as a function of concentration at 10 atm.

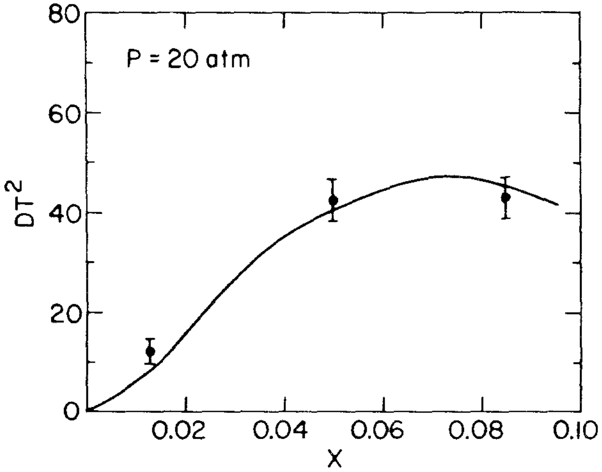


Fig. 17. Comparison of theory and experiment$^{(7)}$ for spin diffusion as a function of concentration at 20 atm.

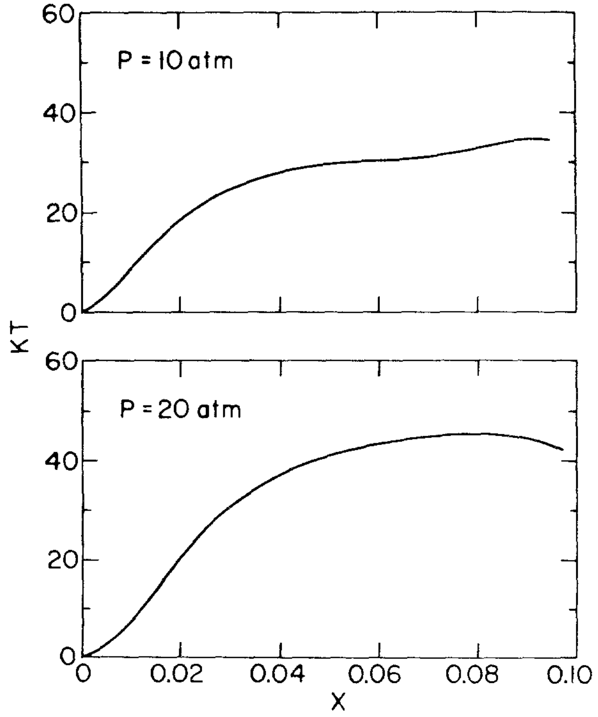


Fig. 18. Theoretical results for thermal conductivity as a function of concentration at 10 and 20 atm.

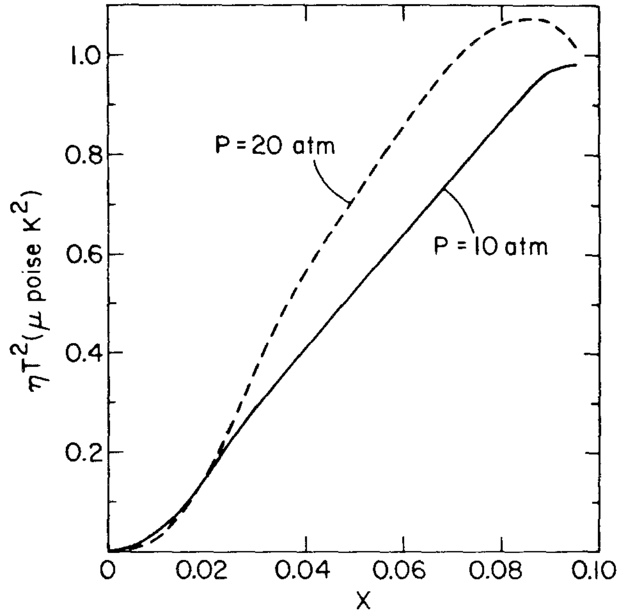


Fig. 19. Theoretical results for viscosity as a function of concentration at 10 and 20 atm.

We have also calculated an upper bound for the superfluid transition temperature for ^{3}He quasiparticles in dilute mixtures at pressures of 10 and 20 atm. At both pressures we find the maximum value of T_c occurs at concentrations of ~0.6%; it is 4×10^{-9} K at 10 atm, and 2×10^{-8} K at 20 atm. Thus the application of pressure does not significantly improve the prospects for observing a transition to the superfluid state.

5. DISCUSSION AND CONCLUSION

In determining the concentration and momentum-dependent pseudopotentials which describe the effective interactions between ^{3}He quasiparticles, and between these quasiparticles and the background ^{4}He atoms in dilute mixtures, we have proceeded at two distinct levels. On a macroscopic level we have combined our physical picture of the long wavelength quantities, $V_0^{\uparrow\uparrow}$, $V_0^{\uparrow\downarrow}$, and u_0, as spatial averages of configuration space pseudopotentials, with an ansatz for the macroscopic Fermi liquid parameter, $f_0^{\uparrow\downarrow}$, to obtain both a physical understanding and a quantitative account of the variation of these quantities, and of $f_0^{\uparrow\uparrow}$, with concentration and pressure. In this way we have been able to demonstrate the existence of a hierarchy of physical effects which determine these macroscopic quantities in dilute mixtures: interaction-induced short-range correlations, the dominant physical feature, reduce the spatial average of the bare interaction to that observed, say, for the effective interaction f_0^s, between ^{4}He atoms; when a ^{3}He atom replaces a ^{4}He atom, its greater zero point motion gives rise to an enhanced ^{3}He–^{4}He interaction u_0, an enhancement which increases with concentration, x; this greater zero point motion is responsible as well for the further enhancement of the effective interaction between antiparallel spin ^{3}He quasiparticles, $V_0^{\uparrow\downarrow}$; lowest in the hierarchy are the Pauli principle "statistical" correlations, which enhance $V_0^{\uparrow\uparrow}$ by a factor $\sim\alpha^2$ compared to $V_0^{\uparrow\downarrow}$, an effect which decreases with increasing concentration. A similar interplay between zero point motion, the Pauli principle, and the interaction-induced correlations is responsible for the difference between the direct effective interaction, $V_0^{\uparrow\downarrow}$, and its counterpart in pure ^{3}He, $f_0^{\uparrow\downarrow}$, as well as the approximate equality of $f_0^{\uparrow\uparrow}(^3\text{He})$ and $V_0^{\uparrow\uparrow}$ in systems at the same density. Because the Pauli enhancement of $V_0^{\uparrow\uparrow}$ is maximum in the low concentration limit, we are able to understand why f_0^a is positive in this limit, decreases with increasing x, and becomes negative for pure ^{3}He.

This same hierarchy of physical effects is responsible for the results we obtain at the microscopic level for the changes in the momentum dependence of the effective interactions with concentration and pressure. By making the ansatz that the short-range part of effective interactions between ^{3}He and ^{4}He quasiparticles, and that between ^{3}He atoms of antiparallel spin, is identical to that between ^{4}He atoms at the same system density, we have a unique prescription for u_q and $V_q^{\uparrow\downarrow}$; and, since $\chi(q, 0)$ is known from the work of Aldrich and Pines, for $f_q^{\uparrow\downarrow}$. We are thus able to reduce the number of free parameters which determine the ^{3}He quasiparticle interactions to a single free parameter, δ, the difference in the range of the repulsive part of the interaction between parallel spin quasiparticles compared to that of antiparallel spin quasiparticles. Once δ is determined by our "best fit" scattering amplitude calculation of the spin diffusion, there are no free parameters in the theory.

Our work on dilute mixtures illuminates the choice made by Aldrich and Pines for the quasiparticle pseudopotentials in pure ^{3}He. Because the bare interaction between He atoms changes sign at a distance (2.68 Å) comparable to that which characterizes the amplitude of zero point motion of ^{3}He atoms, when one replaces a ^{4}He atom by a ^{3}He atom in the liquid, its enhanced zero point motion changes dramatically the nature of the interaction (from attractive to repulsive) at distances $2.68 \lesssim r \lesssim r_c$. The dependence of r_c on concentration and pressure we find in the mixture leads naturally to the value adopted by Bedell *et al.*$^{(18)}$ for pure ^{3}He at 21 atm ($r_c \cong 2.82$ Å), and by AP for ^{3}He at SVP ($r_c \cong 3$ Å). Moreover, as one might expect, spin correlations influence somewhat this zero point enhancement of the range of the repulsive interaction; again, the dependence of $\delta(\equiv r^{\uparrow\uparrow} - r^{\uparrow\downarrow})$ on concentration and pressure we find for the dilute mixtures leads naturally to the results found by Bedell and Pines for pure ^{3}He. Consistent with their influence on the spatial average of the pseudopotentials, spin correlations have a much smaller effect on r_c than does zero point motion.

Our momentum-dependent pseudopotentials, $V_q^{\uparrow\downarrow}$, and u_q, are determined to sufficient accuracy that the resulting effective quasiparticle interaction, $f_q^{\uparrow\downarrow}$, changes sign for $q \lesssim p_F$, as BBP argued it must to explain spin diffusion in dilute mixtures. Our results provide the first physical explanation for this behavior, as well as a detailed model for its concentration dependence and pressure dependence, and the first calculation of the bahevior of $f_q^{\uparrow\uparrow}$ as a function of momentum, concentration, and pressure.

We have reported in this paper on two of the three experimental tests of the accuracy of the pseudopotentials proposed here, and find good agreement between theory and experiment for both the transport properties and the maximum superfluid transition temperature as a function of concentration and pressure. The shift with concentration and temperature of the phonon–maxon–roton spectrum of dilute mixtures provides a further direct test of u_q; as we show elsewhere,(15) the excellent agreement between our theoretical calculations and the neutron scattering measurements of this quantity gives us considerable confidence in our model for u_q.

Overall we expect that for a given choice of Landau parameters, our pseudopotentials are accurate to some 10%; improved measurements of the Landau parameters as a function of concentration and pressure will test the simple scaling laws we have proposed for these quantities, while careful measurements of the transport properties should test our predicted maxima in these quantities at concentrations ~4%. Since at present there is considerable uncertainty in the experimental determination of the ^{3}He quasiparticle effective mass as a function of concentration and pressure,(32) and in the role which slip plays in the measured mixture viscosities,(28) further experimental work to resolve these discrepancies would seem highly desirable.

ACKNOWLEDGMENTS

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To be invited to contribute to a volume which honors a theoretical physicist of the caliber of Ilya M. Lifshitz is both a privilege and responsibility. It is a privilege because Ilya Mihaelovitch was one of the great physicists of our time; a responsibility because his personal standards for scientific achievement were of the highest. Those among us who had the opportunity to know him well remember not only his keen intellect, remarkable physical intuition, and superb taste in choosing a research problem, but also his quick wit, his personal warmth, and his wonderful generosity of spirit. Had this been a festschrift, because of Ilya's continuing interest in the properties of quantum liquids, we would have hoped that this contribution would bring him pleasure. We dedicate this paper to his memory.

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